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MATHEMATICAL METHODS
UNITS 3 AND FOUR

Common Assessment Task 1: Investigative Project

THEME: **MAXIMA AND MINIMA**

STARTING POINT 1

x^{α} and beyond

STUDENT NUMBER: [redacted]

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ABSTRACT

The theme of this investigative project was maxima and minima. For starting point one this was restricted to the families of curves with the equations: $y=x^{\alpha}$ and $y=x^{\alpha}\pm x^{\beta}$. The objective of the project is to explore these graphs and to answer the specific questions posed.

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CONTENTS

THE SOLUTION PROCESS

The main focus of this investigation was on the theme maxima and minima. For starting point 1 this was confined to graphs of the form $y=x^{\alpha}$ at the beginning, and later extended to cover graphs of the equation $y=x^{\alpha}\pm x^{\beta}$.
For the purposes of this investigation:
'Even' numbers are the set of (2n), where n can be any integer, or zero.
'Odd' numbers are the set of (2n+1), where n can be any integer, or zero.
To improve legibility, equations of the format $y=x^{-¾}$ have been written as $y=x^{-3/4}$.
'Fraction' is taken to mean any simplified fraction with integers in the numerator and denominator, with a value between 0 and 1, or -1 and 0 (not inclusive), except where otherwise stated.

PART A

(i)

The first part of the project required the determination of the values of α in the equation $y=x^{\alpha}$ for which the curve has a minimum at x=0, and α is a positive integer.
$\therefore\ \alpha \in J^{+}$.
The graphs sketched were $y=x^1$, $y=x^2$, $y=x^3$, $y=x^4$, $y=x^5$, $y=x^{20}$, and $y=x^{21}$.
It was found that all the graphs where α was an even number resembled a parabolic curve. However, the graphs were *not* parabolas. The difference was in the gradient (m). All of these graphs had 1 minimum at x=0.

A turning point is one example of a stationary point. It is where the curve 'doubles back on itself.' It can also be defined as the point where the value of y reaches a maximum or minimum, whether local or absolute. These stationary points require the gradient to be zero at one instant, to enable them to go from a negative gradient to a positive one, or vice versa. For this reason they can be recognised from the derivative curve as the point at which the graph *cuts* the x-axis.

eg. $f(x)=x^4$
$f'(x)=4x^3$
at x=0
$f'(0)=4\times 0^3=4\times 0=0$
Therefore the stationary point occurs at x=0 (see following page).

All the graphs of the formula: $y=x^{\alpha}$, where $\alpha=2n$, and $n \in J^{+}$ have a minimum at x=0 (see following page).

As several graphs were sketched it was assumed that the trend would hold for all graphs of the suggested format.

(ii)

It was assumed that this section was also only designed to deal with positive integers. It was required to find for what values of α a point of inflexion occurs at x=0. All the graphs where α was an odd number, except for $\alpha=1$, resembled the equation $y=x^3$, as seen from the graphs on the following 2 pages. These graphs (except y=x) all had a *stationary* point of inflexion at x=0.

A point of inflexion occurs when a curve changes the direction of its concavity, and crosses its tangent without doubling back on itself (see Diagram 1). This can be shown on a graph of the derivative (which is the gradient function) where a turning point occurs.

Diagram 1. A point of inflexion.

A stationary point of inflexion occurs where the curve actually reaches zero gradient, but the gradient then goes back to its original direction.

eg. The sequence of the gradient could be:
Positive → Zero → Positive.
or Negative → Zero → Negative.

A stationary point of inflexion can be detected by the derivative graph where a 'bounce-point' on the x-axis occurs. This is a turning point that touches but does not cut the axis.

eg. $f(x)=x^5$

$f'(x)=5x^4$

at x=0

$f'(x)=5\times 0^4=5\times 0=0$

Therefore the stationary point occurs at x=0 (see page 7).

The graph of y=x did not fit the above trend. It had the same domain and range, but did not have a point of inflexion.

All the graphs of the formula: $y=x^{\alpha}$, where $\alpha=2n+1$, and $n \in J^+$, have a stationary point of inflexion at x=0 (see page 7).

As several graphs were sketched, it was assumed that the feature of having a point of inflexion at x=0 would be applicable to all graphs of the proposed formula.

PART B

(i)

For this section positive reciprocal values of α were investigated. Graphs were sketched of $y=x^{\alpha}$, for α equals the reciprocals of 2, 3, 4, 5, 20 and 21 (see the following 2 pages). These graphs are similar to the inverses of: $y=x^2$, $y=x^3$, $y=x^4$, $y=x^5$, $y=x^{20}$ and $y=x^{21}$ respectively.
This would mean that they have been reflected over the line y=x. However, for the reciprocals of positive, even numbers, the reflection is 'incomplete' as only the positive y-ordinates are shown.

For the equation: $y=x^{\alpha}$:

Where α is the reciprocal of a positive even integer, the domain is $[0, \infty)$. (see Table 1).
Where α is the reciprocal of a positive odd integer, the domain is $(-\infty, \infty)$. (see Table 1).

The tangent at x=0 is 'undefined', as the tangent is parallel to the y-axis. This phenomenon can be seen clearly from the graph of the first derivative. As the derivative curve approaches x=0 it approaches its asymptote of the y-axis, and ipso facto it may never actually reach the y-axis, or have a (real number) value at x=0.

A minimum occurs where α is the reciprocal of a positive even integer.

(ii)

For this section other positive, fractional values of α were investigated, eg ¾.
The domain and range of these functions were as follows:

NUMERATOR	DENOMINATOR	DOMAIN	RANGE
Odd integers	Odd integers	(-∞, ∞)	(-∞, ∞)
Even integers	Odd integers	(-∞, ∞)	[0, ∞)
Odd integers	Even integers	[0, ∞)	[0, ∞)
Even integers	Even integers	see text	[0, ∞)

Table 1.

Note that a numerator of 1 follows the trend of the other odd numbers.
These graphs (except x to the power of an unsimplified fraction (even on even)) are shown on the following 3 pages.
When x is raised to the power of a fraction that is an even number over an even number, the domain will vary as explained below. This situation is not applicable to most aspects of mathematics, as it is a mathematical convention to simplify all fractions being used. In certain combinations the effect of raising x to an unsimplified fraction is to extend the domain.

Consider the fraction three fifths (odd on odd):
The order of operations does not matter.

eg. $x=-32$
$y=(x^3)^{1/5}=-32768^{1/5}=-8$
$y=(x^{1/5})^3=-2^3=-8$

Consider the fraction two thirds (even on odd):
The order of operations does not matter.

eg. $x=-8$
$y=(x^2)^{1/3}=64^{1/3}=4$
$y=(x^{1/3})^2=-2^2=4$

Consider the fraction three quarters (odd on even):
The order of operations does not matter.

eg. $x=16$
$y=(x^3)^{1/4}=4096^{1/4}=8$
$y=(x^{1/4})^3=2^3=8$

Consider the fraction two quarters (even on even):
The order of operations does matter.

eg. $x=-4$
$y=(x^2)^{1/4}=16^{1/4}=2$
$y=(x^{1/4})^2=$undefined

but when α is simplified to ½:
$y=x^{1/2}=(-4)^{1/2}=$undefined

eg. $x=-9$
$y=(x^2)^{1/4}=81^{1/4}=3$
$y=(x^{1/4})^2=$undefined

but when α is simplified to ½:
$y=x^{1/2}=(-9)^{1/2}=$undefined

It has been assumed for the remainder of this investigation that all fractional values will be simplified before the values of y are calculated, as this is the mathematical convention.
A similar convention does not permit a fraction where negative integers are in both the numerator and the denominator.
The above assumption means that the order of operation as shown in the examples does not influence the value of y.

For the equation: $y=x^{\alpha}$:
When the numerator and denominator of the fraction α are both odd, then the curve has the basic shape of that on page 12. This has a point of inflexion at x=0, and occupies quadrants 1 and 3.
When the numerator of the fraction α is even and the denominator is odd, then the curve has the basic shape of that on page 13. This has a '*cusp*' at x=0 (see Part C (ii)), and occupies quadrants 1 and 2.
When the numerator of the fraction α is odd and the denominator is even, then the curve has the basic shape of that on page 14. This has a minimum at x=0, and occupies quadrant 1 only.

The curves with a minimum at x=0 are those where the numerator of the index is odd but the denominator of the index is even.
eg. $y=x^{1/2}$, $y=x^{1/4}$, $y=x^{3/4}$, $y=x^{1/6}$, $y=x^{3/6}$, $y=x^{5/6}$, $y=x^{1/8}$, etc.

For $f(x)=x^{1/2}$:
$f'(x)=\frac{1}{2}x^{-1/2}$
at x=0
$f'(x)=\frac{1}{2}\times0^{-1/2}$=undefined (as you cannot divide by zero)

For $f(x)=x^{3/4}$:
$f'(x)=\frac{3}{4}x^{-1/4}$
at x=0
$f'(x)=\frac{3}{4}\times0^{-1/4}$=undefined (as you cannot divide by zero)

This means the gradient at x=0 where the minimum occurs is undefined, and the tangent is parallel to the y-axis. The minimum can be found in this case from the asymptote on the derivative curve, however this method is not conventional. What is conventional is to look for the point where the derivative cuts the x-axis, and this always indicates a turning point (as m=0). As the asymptote on the derivative curve does not indicate a turning point, but undefined gradient, it can be said that the minimum at x=0 is not detected by condition $f'(x)=0$.

PART C: EXTENSION

(i)

For this section of the investigation, it was required to investigate the effect(s) of adding and subtracting functions to turning points and points of inflexion. The function had to be of the form: $y=x^{\alpha}\pm x^{\beta}$, where $y=x^{\alpha}$ was the original curve. Integer values of α and β were investigated first.

Many graphs were sketched to enable general rules to be found. The results are listed below in Table 2: Turning point graphs (ie. functions with turning points $\pm$ $y=x^{\beta}$), and Table 3: Inflexion graphs (ie. functions with turning points $\pm$ $y=x^{\beta}$). For the actual sketches see Appendices 1 and 2 respectively.

α	+ or -	β	Turning Points	Points of Inflexion	
				Stationary	Non-stationary
2	+	-2	2	0	0
4	+	-2	2	0	0
2	-	-2	0	0	2
4	-	-2	0	0	2
2	+	-1	1	0	2
4	+	-1	1	0	2
2	-	-1	1	0	1
4	-	-1	1	0	1
2	+	1	1	0	0
4	+	1	1	0	0
2	-	1	1	0	0
4	-	1	1	0	0
2	+	3	2	0	1
4	+	5	2	0	1
2	-	3	2	0	1
4	-	5	2	0	1
4	+	3	1	1	1
6	+	5	1	1	1
4	-	3	1	1	1
6	-	5	1	1	1
2	+	4	1	0	0
2	-	4	3	0	2
4	-	2	3	0	2

Table 2. Turning point graphs.

From the results in Table 2, the following conclusions were drawn:

$x^{+even} + x^{-even} \rightarrow$	2 turning points
$x^{+even} - x^{-even} \rightarrow$	2 points of inflexion
$x^{+even} + x^{-odd} \rightarrow$	1 turning point & 2 points of inflexion
$x^{+even} - x^{-odd} \rightarrow$	1 turning point & 1 point of inflexion
$x^{+even} \pm x \rightarrow$	1 turning point
$x^{+even} \pm x^{+lesser\ odd} \rightarrow$	1 turning point, 1 stationary point of inflexion & 1 point of inflexion
$x^{+even} \pm x^{+greater\ odd} \rightarrow$	2 turning points & 1 point of inflexion
$x^{+even} + x^{+even} \rightarrow$	1 turning point
$x^{+even} \pm x^{+even} \rightarrow$	3 turning points & 2 points of inflexion

Note that points of inflexion refer only to the non-stationary variety when used here.

α	**+ or -**	**β**	**Turning Points**	**Points of Inflexion**	
				Stationary	**Non-stationary**
3	+	-2	1	0	1
5	+	-2	1	0	1
3	-	-2	1	0	1
5	-	-2	1	0	1
3	+	-1	2	0	0
5	+	-1	2	0	0
3	-	-1	0	0	2
5	-	-1	0	0	2
3	+	1	0	0	1
5	+	1	0	0	1
3	-	1	2	0	1
5	-	1	2	0	1
3	+	2	2	0	1
5	+	2	2	0	1
3	-	2	2	0	1
5	-	2	2	0	1
3	+	6	1	1	1
5	+	6	1	1	1
3	-	6	1	1	1
5	-	6	1	1	1
3	+	5	0	0	1
5	-	3	2	1	2
3	-	5	2	1	2

Table 3. Inflexion graphs.

[redacted]

From the results in Table 3, the following conclusions can be drawn:

$x^{+odd} \pm x^{-even} \rightarrow$	1 turning point & 1 point of inflexion
$x^{+odd} + x^{-odd} \rightarrow$	2 turning points
$x^{+odd} - x^{-odd} \rightarrow$	2 point of inflexions
$x^{+odd} + x \rightarrow$	1 point of inflexion
$x^{+odd} - x \rightarrow$	2 turning points & 1 point of inflexion
$x^{+odd} \pm x^{+lesser\ even} \rightarrow$	2 turning points & point of inflexion
$x^{+odd} \pm x^{+greater\ even} \rightarrow$	1 turning point, 1 stationary point of inflexion & 1 point of inflexion
$x^{+odd} + x^{+odd} \rightarrow$	1 point of inflexion
$x^{+odd} \pm x^{+odd} \rightarrow$	2 turning points, 1 stationary point of inflexion & 2 points of inflexion

x^{+odd} refers to graphs where the index can be any positive, odd, integer other than 1. Also, points of inflexion refer only to the non-stationary variety when used here.

It is clear from the data collected and the conclusions drawn that maxima, minima and points of inflexion can be created and destroyed simply by adding the function $y=x^{\beta}$ to the original, $y=x^{\alpha}$.

eg. An absolute minimum is destroyed in the function $y=x^2-x^{-2}$, and 2 points of inflexion are created (see page 18).

eg. Three maxima and minima, and 2 points of inflexion are created by the function $y=x^2-x^4$, while the original absolute minimum is destroyed (see page 19).

eg. A stationary point of inflexion is destroyed by the function $y=x^5+x^{-1}$, but 2 turning points are created (see page 20).

The full set of possible combinations of non-integer indices is enormous, and hence beyond the scope of this investigation. However, several interesting examples are shown below, and sketched on the following pages.

The graph of $y=x^{+1/odd}$ has a point of inflexion at x=0, where the tangent is parallel to the y-axis.

$A(x)=x^{1/3}-x$
The point of inflexion is still at x=0, but 2 turning points have been created.

$B(x)=x^{1/3}+x^2$
The point of inflexion is still at x=0, but 1 turning point and 1 other point of inflexion have been created.

$C(x)=x^{1/3}+x^3$
The point of inflexion is still at x=0, but 2 other points of inflexion have been created.

The derivative curve has been included on these graphs to improve the ease with which the various features of the original function can be recognised. As explained previously, the gradient, or derivative, curve can be used to detect turning points and points of inflexion.

Ordinarily, points of inflexion are detected from the point on the derivative curve at which there is a turning point. When this point is on the x-axis, then a stationary point of inflexion occurs on the original function. It should be understood that the value of the y-ordinate in the turning point of the derivative has no immediate bearing on the position at which the inflexion point occurs on the function, only the x-ordinates are identical.

An important exception to the rules above is the point of inflexion (and all stationary points) where the gradient is undefined. For these graphs, eg. $y=x^{1/3}$, the derivative curve is 'undefined' for that value of x. This means that the gradient function reaches an asymptote at this point. However, the point of inflexion still exists for all this, as even if it is not detected mathematically, it can be shown graphically that it alters the direction of its concavity, and crosses its tangent without doubling back on itself (see Diagram 1).

(ii)

For this section of the project it was required to investigate graphs with the property of a '*cusp*' at x=0. Subsequently it was required to find the result of adding the functions y=x and $y=x^2$ to functions with cusps at x=0. Lastly it was required to find the equation of the curve with cusps at x=0 and x=1.

Diagram 2. The general shape of a cusp.

As the definition of what constituted a cusp was equivocal, some assumptions were made. It was assumed that the cusp had to have a concavity as shown in Diagram 2. It was also assumed that the cusp did not have to be symmetrical, as it was possible to have 1 cusp with 1 point of inflexion.
The final assumption for this section was that the curve with cusps at x=0 and x=1 did *not* have to have an equation of the form $y=x^{\alpha}\pm x^{\beta}$, as this was not specified in the question.

To begin with, some equations involving absolute values were found to contain cusps.
eg.
$y=|\sin x|$, $y=|\mathrm{Tan}^{-1} x|$, $y=\sqrt{|x|}$, (see Appendix 3).
However, as these graphs are not of the prescribed format, they were not valid.

From the information gathered during Part B (ii) of this investigation, several curves that appear to exhibit the features of a cusp were identified.
eg. $y=x^{2/3}$, $y=x^{2/5}$, $y=x^{4/5}$, $y=x^{2/7}$, $y=x^{4/7}$, $y=x^{6/7}$, etc.
These graphs are sketched on the following page.

The trend that emerges is that all graphs with the equation $y=x^{\alpha}$, where the fraction α has a positive even numerator and a positive odd denominator, have a cusp at x=0.

The gradient at the cusp is undefined, as on the graphs the derivative curve reaches an asymptote at x=0.

eg. $A(x)=x^{2/3}$
$A'(x)={}^{2}/_{3}x^{-1/3}$
at x=0
$A'(x)={}^{2}/_{3}\times 0^{-1/3}$=undefined (as you can not divide by zero)

Another interesting feature of the graphs is that the smaller the value of α, the more curved the graph is between x=-1 and x=1, and the closer the value of α to 1, the straighter it is in each quadrant - it approaches the function $y=|x|$.

The functions y=x and $y=x^2$ have been added to the 'cusp-functions', $y=x^{2/5}$ and $y=x^{4/5}$ on pages 29 and 30.

When y=x was added, 1 turning point was created. The cusp at x=0 was not destroyed (see derivative curve). As $x \to \pm\infty$, the curve approached y=x.

When $y=x^2$ was added, the cusp remained at x=0 (see derivative curve), and 2 points of inflexion were created. As $x \to \pm\infty$, the curve approached $y=x^2$.

The curve devised with cusps at x=0 and x=1, was found from knowledge of translation of graphs and addition of ordinates. The equation of the graph was:
$y=x^{2/3}+(x-1)^{2/3}$ (see page 31).
This was in effect the addition of two functions containing cusps, ie:
$[y=x^{2/3}]$ + $[y=(x-1)^{2/3}]$ $\rightarrow$ $y=x^{2/3}+(x-1)^{2/3}$.
Although the cusps are not at y=0 as the others were, this was assumed to be satisfactory as this was not specified.

The gradient function of the above function now has 2 asymptotes:
One at x=0 and the other at x=1, ie. where the cusps are. The derivative also has an obvious point of inflexion, but this doesn't indicate any turning or inflexion points on the original function.

(iii)

This section required the sketching of any one equation with any combination of features (eg. 1 cusp and 1 point of inflexion). The only restriction was that it must have the form $y=x^{\alpha}\pm x^{\beta}$.

From the equations sketched before this stage, graphs with various unusual properties had already been identified.
The equation: $y=x^{2/5}+x^{3/5}$, is sketched on the following 2 pages.
It has a cusp at x=0, a local maximum (turning point) at $x \approx 0.1$, and a point of inflexion at x=1. To see this from the derivative curve the graph was enlarged.

CONCLUSION

For $y=x^{\alpha}$, when α is a positive, even number, a local minimum occurs at x=0, and when α is a positive, odd number, a stationary point of inflexion occurs at x=0.
A turning point can be detected from the gradient function where it cuts the x-axis, and a point of inflexion where a turning point occurs in the derivative curve.

The tangents to these curves are undefined at x=0, as there is an asymptote in the derivative curve. This is the case where α is any positive fraction.
The domain and range for the function $y=x^{\alpha}$ with positive values of α are given below.

NUMERATOR	DENOMINATOR	DOMAIN	RANGE
Odd integers	Odd integers	$(-\infty, \infty)$	$(-\infty, \infty)$
Even integers	Odd integers	$(-\infty, \infty)$	$[0, \infty)$
Odd integers	Even integers	$[0, \infty)$	$[0, \infty)$
Even integers	Even integers	see Part B (ii)	$[0, \infty)$

NOTE: That this table applies to ***all*** values of α. eg. Fractional values, reciprocals (numerator is odd), and integers (denominator is odd).

It does not matter whether $y=x^{3/4}$ is interpreted as $y=(x^{1/4})^3$ or $y=(x^3)^{1/4}$, and this applies to all simplified fractions. Unsimplified fractions are not considered, as the convention is to simplify.

For positive fractional values of α where the numerator is odd and the denominator is even then the minimum at x=0 is not detected by the derivative curve.

All turning and inflexion points can be created and destroyed by adding or subtracting functions from the original.

When the positive fractional value α has an even numerator and an odd denominator then a cusp occurs at x=0, which has an undefined gradient.

$y=x^{2/3}+(x-1)^{2/3}$ has 1 cusp at x=0 and 1 at x=1.

The graph of $y=x^{2/5}+x^{3/5}$ has a cusp at x=0, a local maximum at $x \approx 0.1$, and a point of inflexion at x=1.

The results are assumed to be accurate due the large quantity of data collected. Further data collection would enable the results to be verified.

A further investigation suggested is to investigate other combinations of α and β in the formula $y=x^{\alpha}\pm x^{\beta}$. eg. Fractional values and/or negative values.
Equations of the form $y=\pm x^{\alpha}\pm x^{\beta}$ could also be investigated.

MATHEMATICAL METHODS

The computer graphing package used was 'Capgraph', put out by the Capricornia Institute in 1987. This graphs functions, the derivative and the second derivative, and prints them out.
Calculus was used to find the equation of the derivative, and to show that the order of operations in Part B (ii) did not matter.
Addition of ordinates was used initially in lieu of the graphing package, and to find the curve with 2 cusps.
Knowledge of translation of curves was also used for that section.
A graphics calculator was used at the start of the investigation.